The compliant pantograph: homogenization and tests

Baptiste Durand[1], Arthur Lebée[1], Pierre Seppecher[2], Karam Sab[1]

[1] Laboratoire Navier (UMR CNRS 8205)
École des Ponts ParisTech; Université Gustave Eiffel; CNRS

[2] Institut de Mathématiques de Toulon,
Université de Toulon, Toulon, France

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ArchiMatHOS

"Architected materials designed with higher-order homogenization"

- Scientific objectives:
  - Derive a predictive homogenization scheme for strain gradient elasto-static continua
  - Generate new microstructures from topology optimization
  - Validate the non-standard behavior from experiments

- Partners:

LABORATOIRE NAVIER

MSME Laboratoire Modélisation et Simulation Multi Echelle

Imath

Institut ÉLIE CARTAN

LEM3 LABORATOIRE D'ÉTUDE DES MICROSTRUCTURES ET DE MÉCANIQUE DES MATÉRIAUX

Long range strain-gradient effects in elasto-statics?

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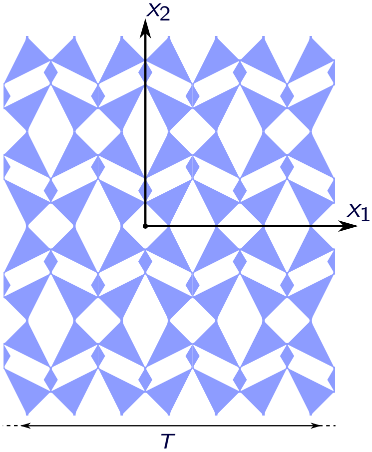
Asymptotic expansion + infinite contrast?

The compliant pantographic microstructure



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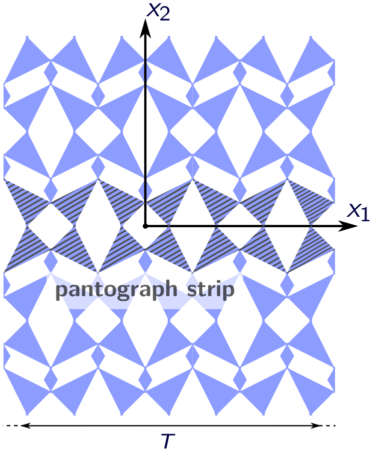

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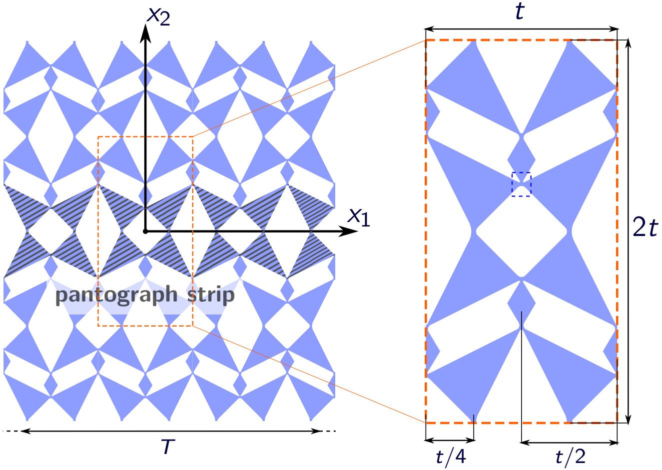

Scale ratio $\frac{t}{T} = \eta \to 0$

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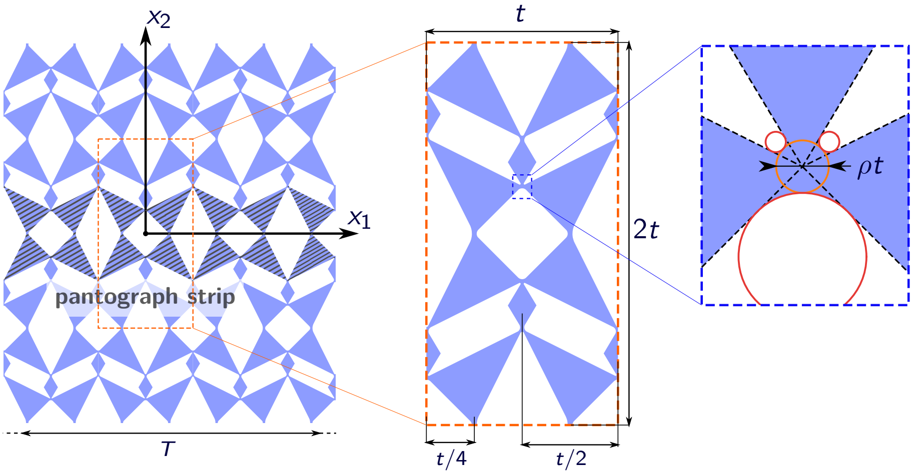

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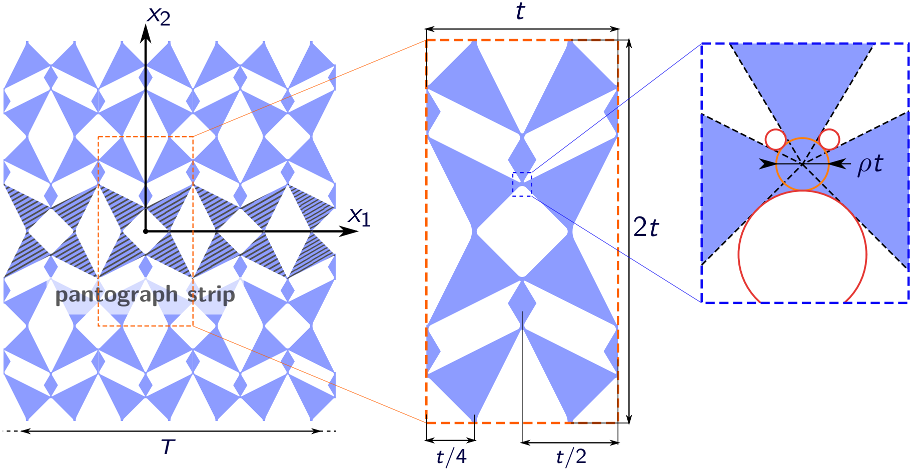
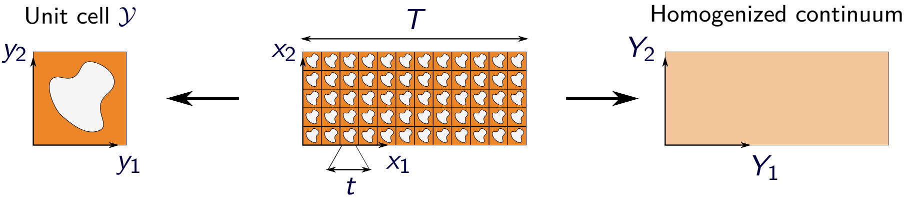


Scale ratio $\frac{t}{T} = \eta \to 0$ Junction thinness $\rho \to 0$

The two scale asymptotic expansion



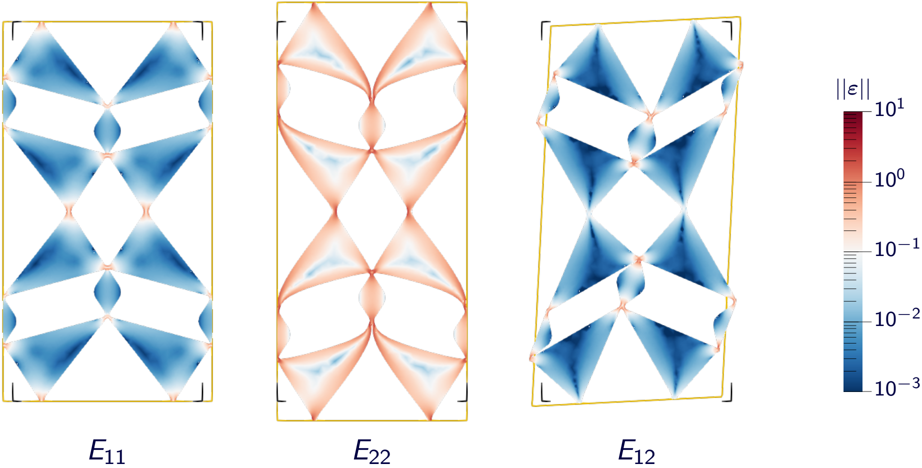
$y_i = \frac{x_i}{t}$ $\quad$ $t \ll T$ $\quad$ $Y_i = \frac{x_i}{T}$

- Scale ratio: $\eta = \frac{t}{T} \ll 1$
- Expansion: $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) = T \sum_{p=0}^{\infty} \eta^p \underline{\boldsymbol{u}}^p(\underline{\boldsymbol{Y}}, \underline{\boldsymbol{y}})$
- Differential operators: $\underline{\boldsymbol{\nabla}}_x = \frac{1}{T}\left(\underline{\boldsymbol{\nabla}}_Y + \frac{1}{\eta}\underline{\boldsymbol{\nabla}}_y\right)$

$\Rightarrow$ Series of **unit-cell problems** loaded by: $\underline{\boldsymbol{U}}(\underline{\boldsymbol{Y}})$, $\boldsymbol{E}(\underline{\boldsymbol{Y}})$, $\boldsymbol{K}(\underline{\boldsymbol{Y}})$...

First-order unit-cell problems

$$\boldsymbol{E} = \underline{\boldsymbol{\nabla}}\,\underline{\boldsymbol{U}}, \quad \underline{\boldsymbol{u}}^{\mathrm{E}}(\underline{\boldsymbol{y}}) = \boldsymbol{E} \cdot \underline{\boldsymbol{y}} + \boldsymbol{h}^{1}(\underline{\boldsymbol{y}}) : \boldsymbol{E}$$



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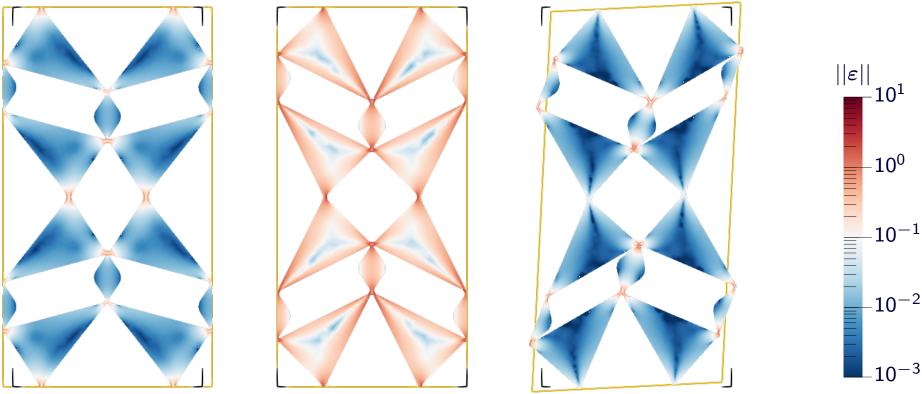

E_{11} E_{22} E_{12}

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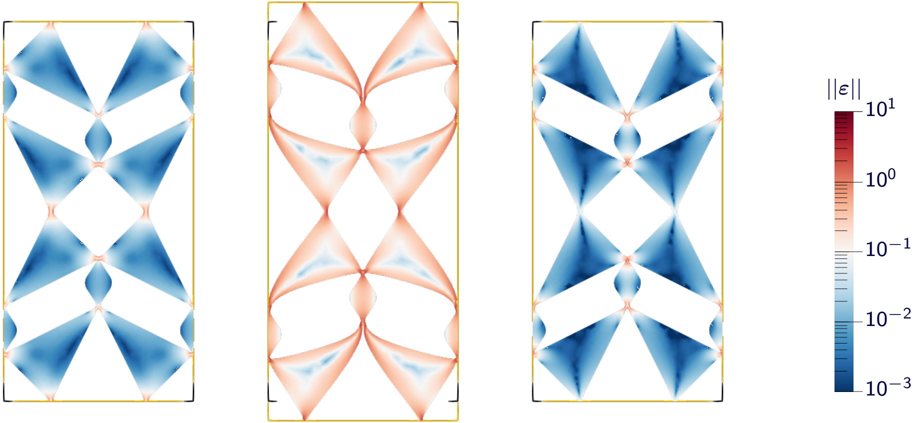

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First-gradient homogenized energy

$$W^{\mathrm{hom},\rho} = \frac{1}{2}\boldsymbol{E} : \boldsymbol{C}^\rho : \boldsymbol{E}$$

$\Rightarrow$ $\boldsymbol{C}^\rho$ is degenerate

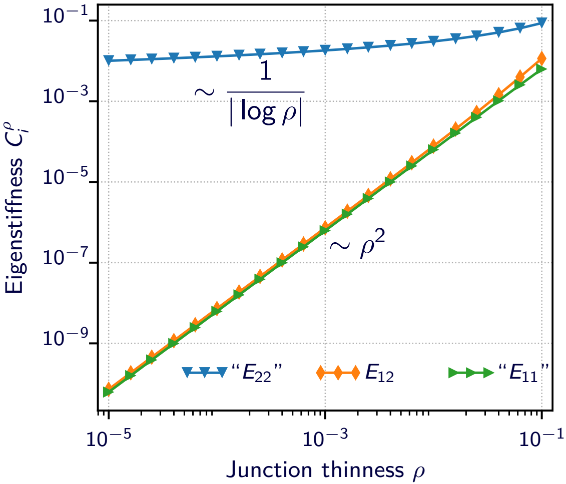
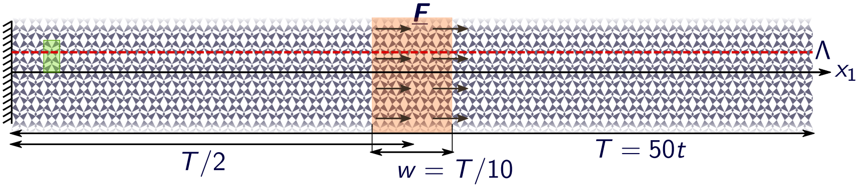


Illustration of strain-gradient effects



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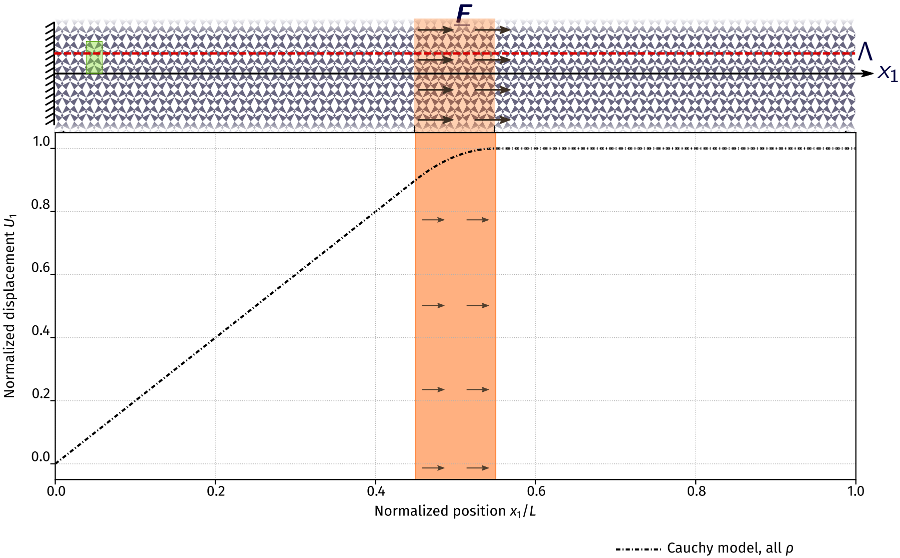

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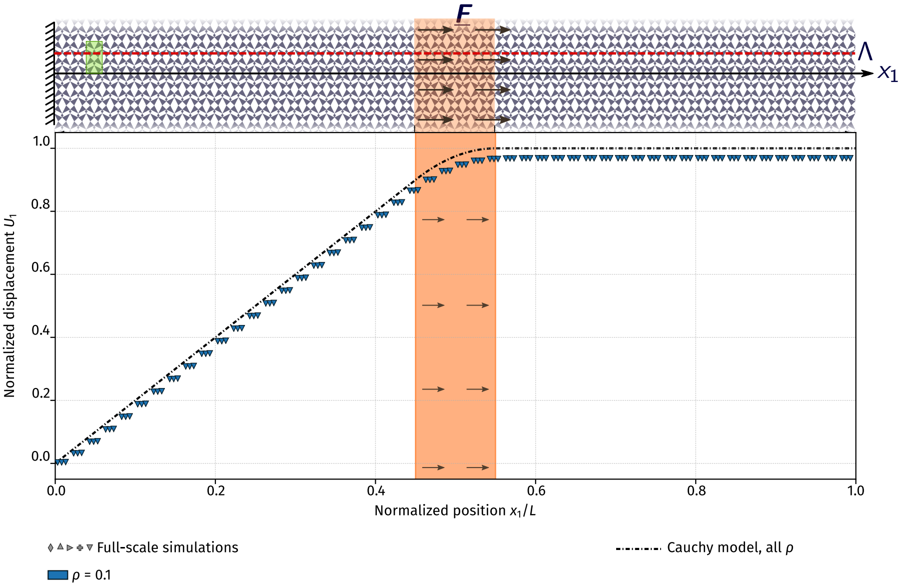

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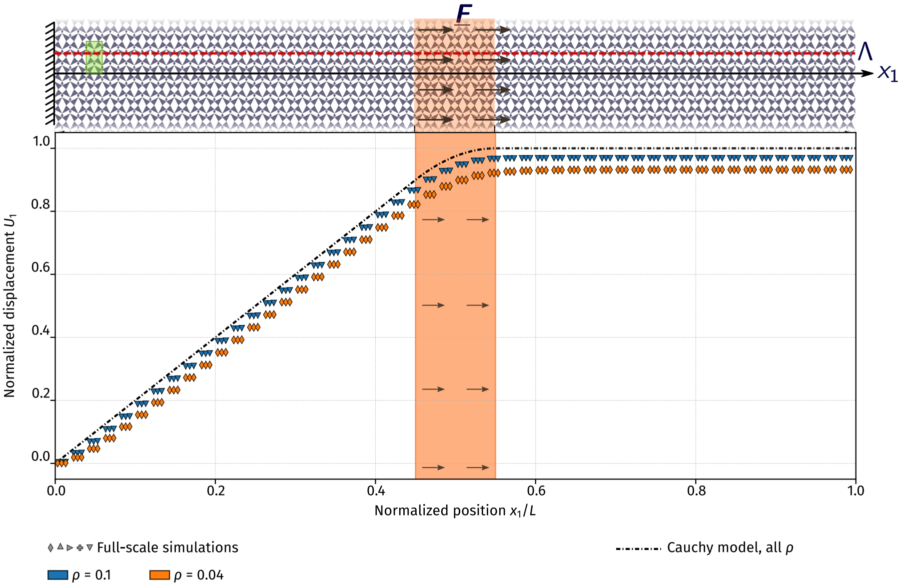

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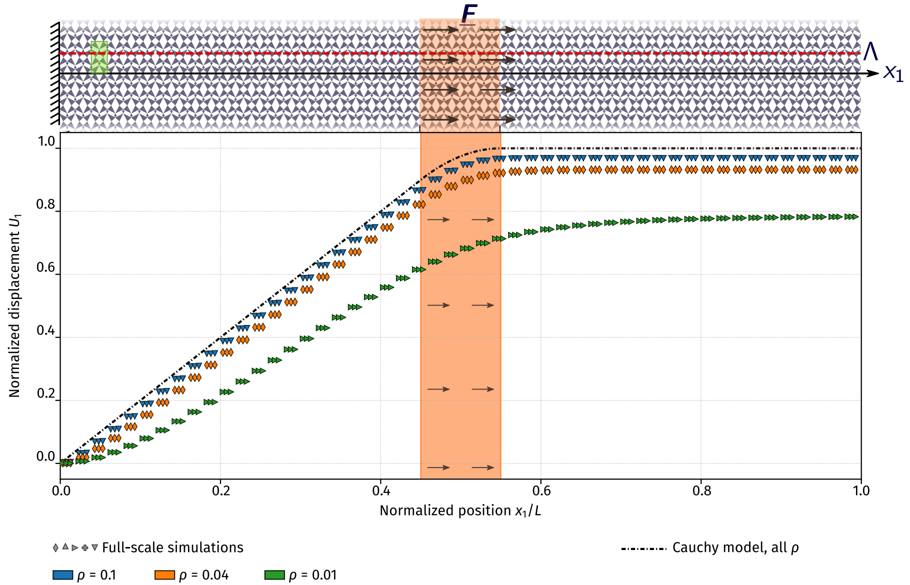

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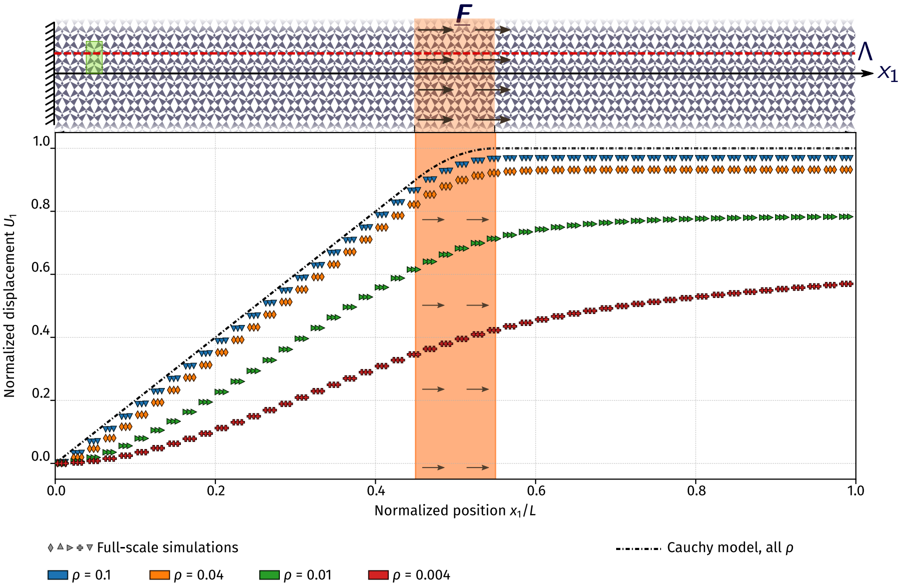

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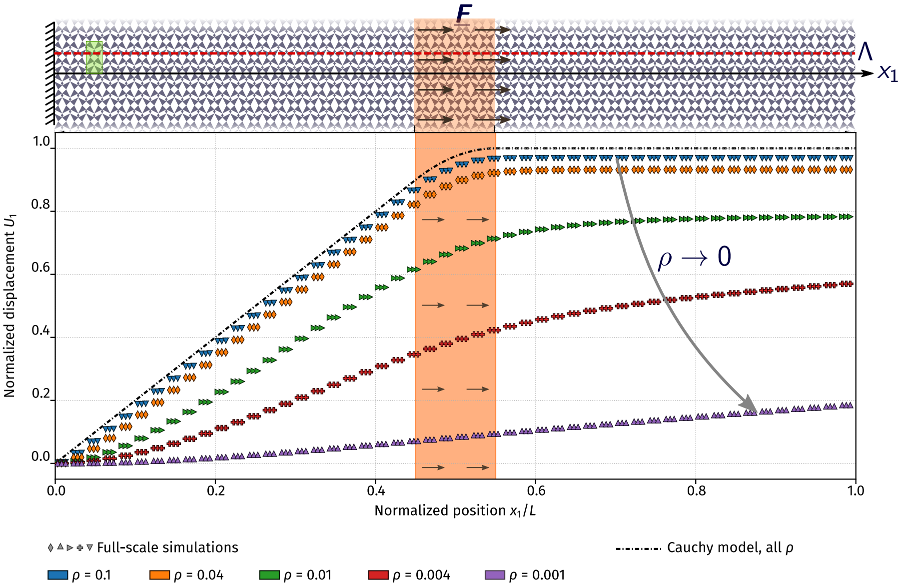
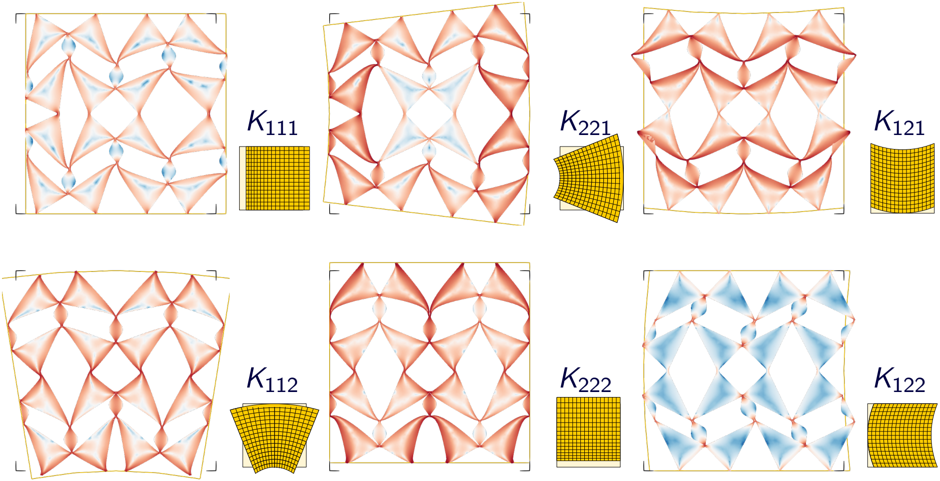


Second-order unit-cell problems

$$K_{\alpha\beta\gamma} = E_{\alpha\beta,\gamma}, \quad \underline{\boldsymbol{u}}^{\mathrm{K}}(\underline{\boldsymbol{y}}) = \frac{1}{2}\widetilde{\boldsymbol{K}} : \left(\underline{\boldsymbol{y}} \otimes \underline{\boldsymbol{y}}\right) + \eta\, \boldsymbol{h}^{1}(\underline{\boldsymbol{y}}) : \left(\boldsymbol{K} \cdot \underline{\boldsymbol{y}}\right) + \eta^{2}\, \boldsymbol{h}^{2}(\underline{\boldsymbol{y}}) \vdots \boldsymbol{K},$$



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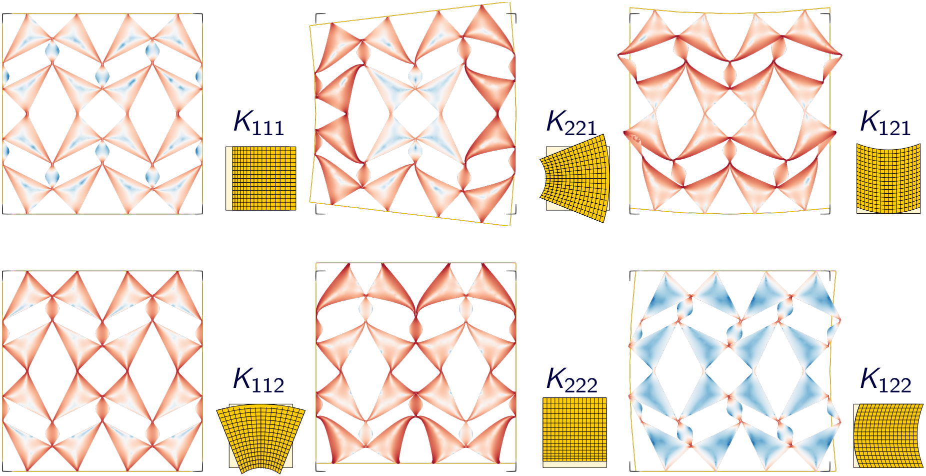

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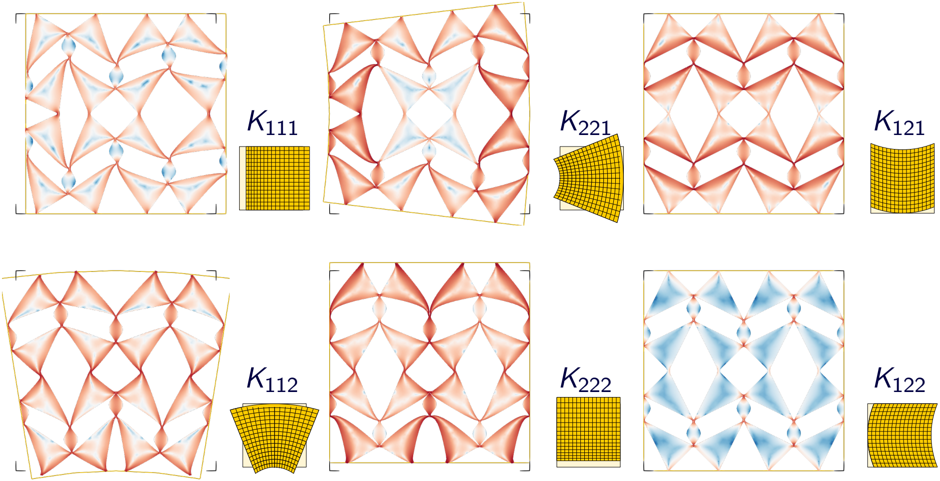

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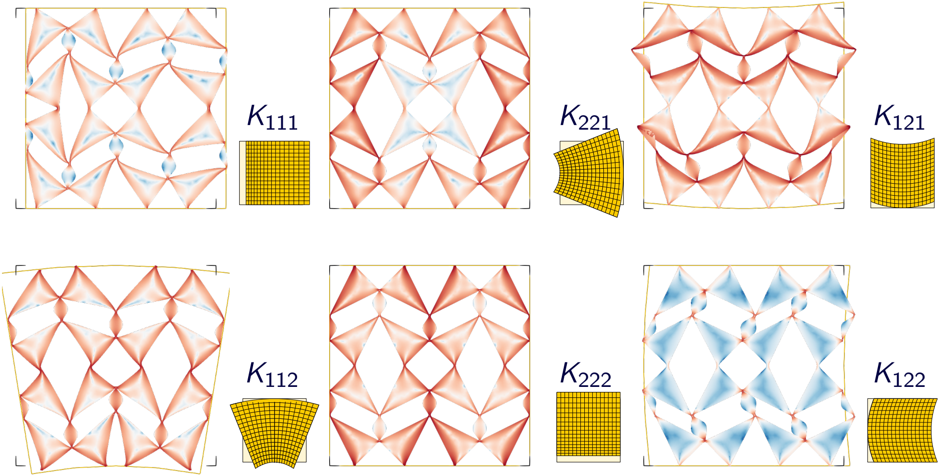

Strain-gradient homogenized energy?

2nd order truncation of displacement expansion:

$$\underline{\boldsymbol{u}}^{\mathsf{K}}(\underline{\boldsymbol{y}}, \underline{\boldsymbol{Y}}) = \underline{\boldsymbol{U}}(\underline{\boldsymbol{Y}}) + \eta\, \boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E}(\underline{\boldsymbol{Y}}) + \eta^2\, \boldsymbol{h}^2(\underline{\boldsymbol{y}}) \vdots \boldsymbol{K}(\underline{\boldsymbol{Y}})$$

Averaged strain energy:

$$W^{\mathrm{hom},\rho} = \frac{1}{2}\left[\boldsymbol{E} : \boldsymbol{C}^\rho : \boldsymbol{E} + \eta^2 \boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^\rho \vdots \boldsymbol{K} + \eta^4 \left(\underline{\boldsymbol{\nabla}}_Y \boldsymbol{K}\right)^{\mathsf{T}} :: \boldsymbol{H}^\rho :: \left(\underline{\boldsymbol{\nabla}}_Y \boldsymbol{K}\right)\right]$$

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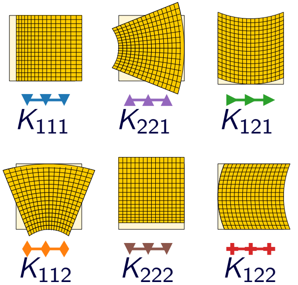
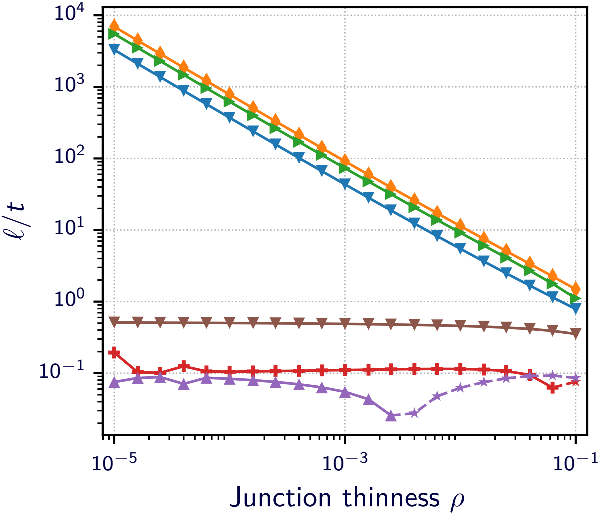
Caveat!

- $\boldsymbol{D}^{\rho}$ not necessarily positive
- $\boldsymbol{H}^{\rho}$ “regularizing”

Retaining significant strain-gradient contributions

$$\ell_{\alpha\beta\gamma} = \sqrt{\frac{D_{\alpha\beta\gamma\gamma\beta\alpha}}{C_{\alpha\beta\beta\alpha}}}$$





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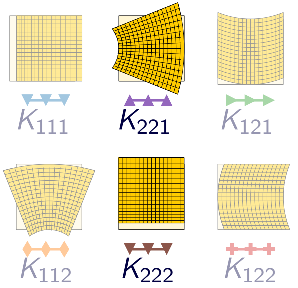

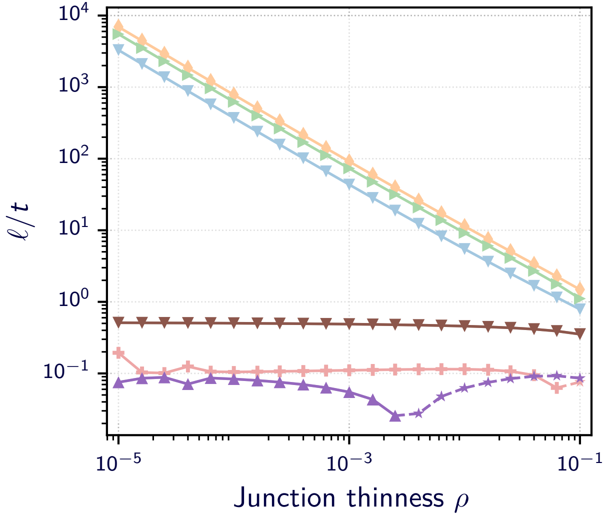

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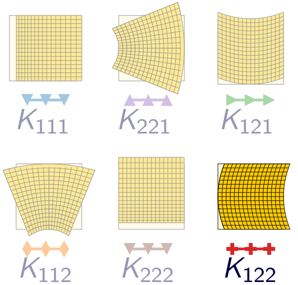

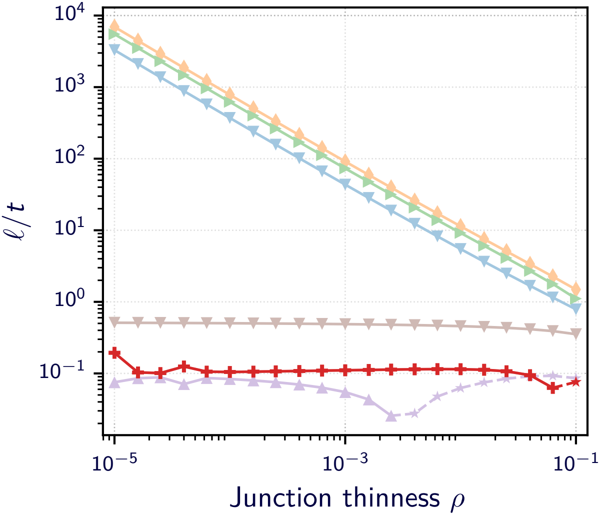

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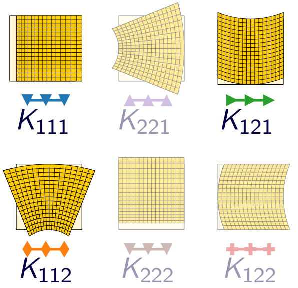
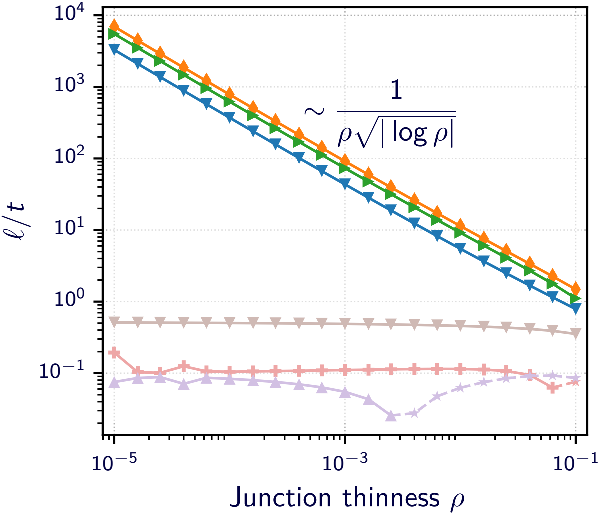


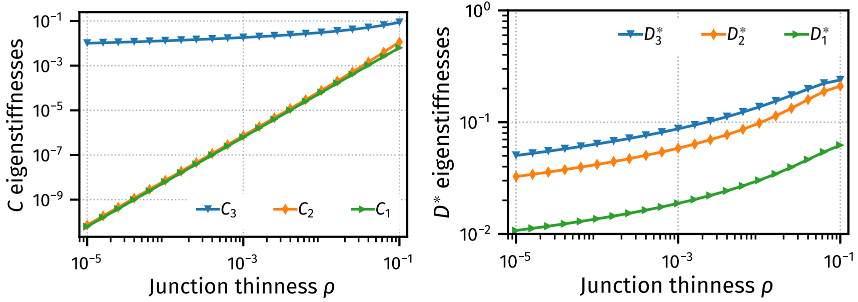


The well-posed strain-gradient homogenized model

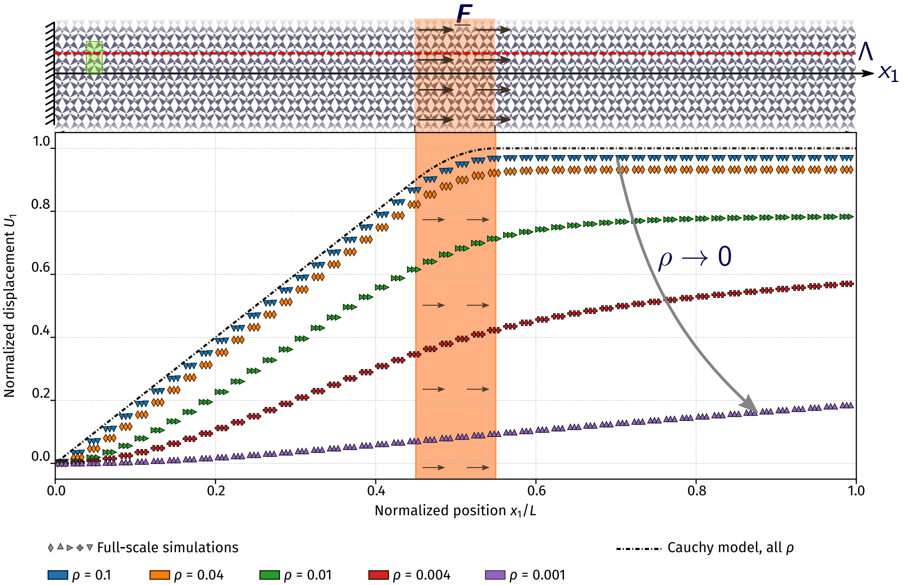
$\boldsymbol{P} =$ projector on span $\{K_{111}, K_{112}, K_{121}\}$, $\qquad \boldsymbol{D}^{*,\rho} = \boldsymbol{P} \vdots \boldsymbol{D}^{\rho} \vdots \boldsymbol{P}$

$\Rightarrow$ **Positive strain-gradient homogenized energy density**

$$W^{\text{hom}}(\boldsymbol{E}, \boldsymbol{K}) = \frac{1}{2}\left(\boldsymbol{E} : \boldsymbol{C}^{\rho} : \boldsymbol{E} + \eta^2 \boldsymbol{K}^{\mathsf{T}} \vdots \boldsymbol{D}^{*,\rho} \vdots \boldsymbol{K}\right)$$



Strain-gradient homogenized model



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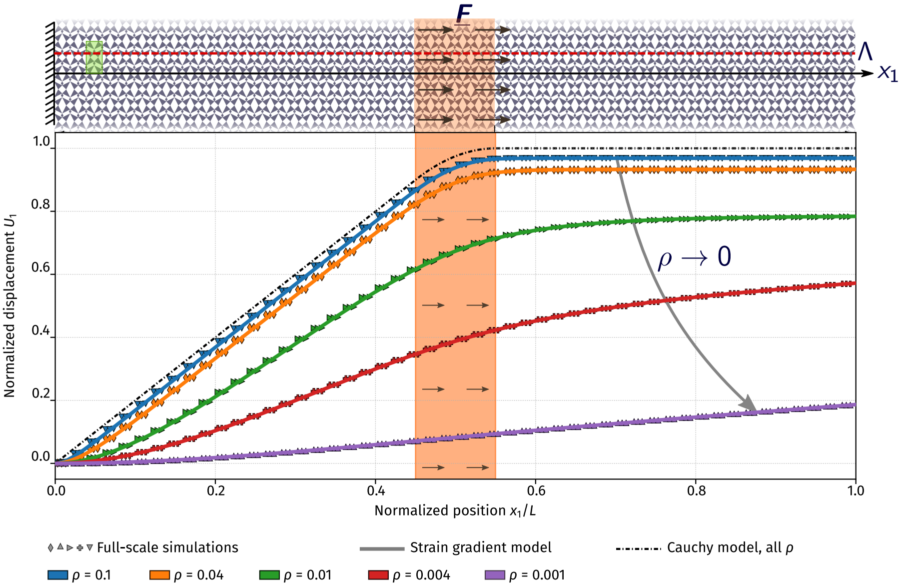
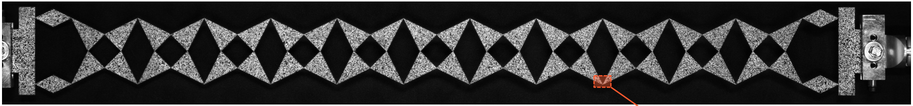
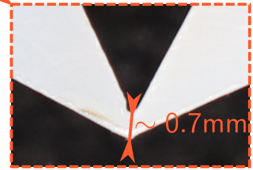
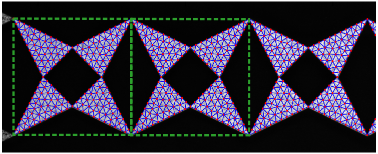


The pantographic strip



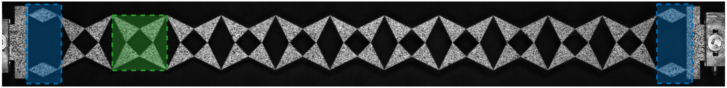


- Laser cut polyoxymethylene (POM), 2mm thick
- $\eta \approx 9\%$ and $\rho \approx 1.1\%$
- DIC averaged on each unit-cell



Results

Weak clamp



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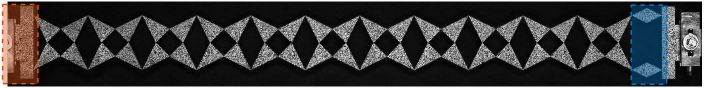

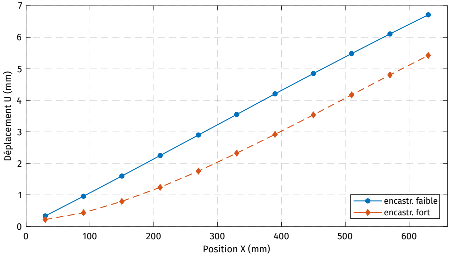

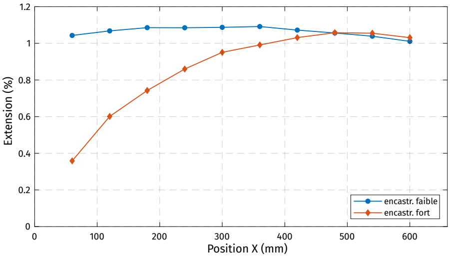

$$E^{\text{hom}} \Big/ E^{\text{POM}} \approx 6 \times 10^{-5} \qquad \text{and} \qquad \ell / t \approx 2.15$$

Conclusion

Main results

- The asymptotic expansion yields predictive strain-gradient moduli *when they are significant*
- Experimental evidence of strain-gradient effects in the linear elasticity framework

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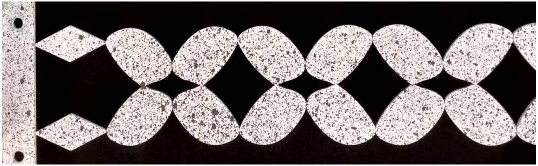

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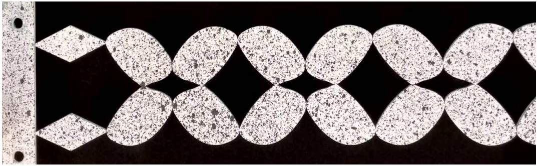

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Outlooks

- Enriched continua?
- Finite deformations?

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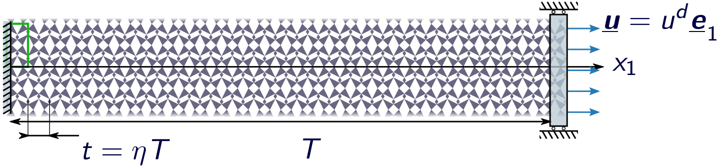
Full scale problem and approximate solutions



- First-gradient model:

$$\underline{\boldsymbol{u}}^{\mathrm{C}}(\underline{\boldsymbol{x}}) = \underline{\boldsymbol{U}}^{\mathrm{C}}(\underline{\boldsymbol{Y}}) + \eta \boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E}^{\mathrm{C}}(\underline{\boldsymbol{Y}})$$

- Strain-gradient model:

$$\underline{\boldsymbol{u}}^{\mathrm{SG}}(\underline{\boldsymbol{x}}) = \underline{\boldsymbol{U}}^{\mathrm{SG}}(\underline{\boldsymbol{Y}}) + \eta \boldsymbol{h}^1(\underline{\boldsymbol{y}}) : \boldsymbol{E}^{\mathrm{SG}}(\underline{\boldsymbol{Y}}) + \eta^2 \boldsymbol{h}^2(\underline{\boldsymbol{y}}) \vdots \boldsymbol{K}^{\mathrm{SG}}(\underline{\boldsymbol{Y}})$$

- Error estimate:

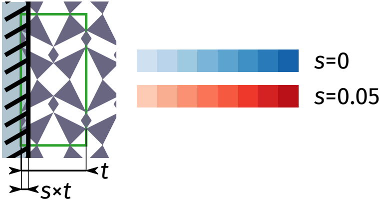
$$\delta(\underline{\boldsymbol{u}}^{\mathrm{X}}) = ||\underline{\boldsymbol{u}}^{\mathrm{fs}} - \underline{\boldsymbol{u}}^{\mathrm{X}}||_2 \Big/ ||\underline{\boldsymbol{u}}^{\mathrm{fs}}||_2$$

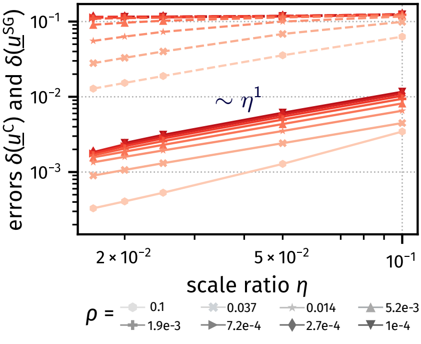
Total displacement error w.r.t full scale simulations

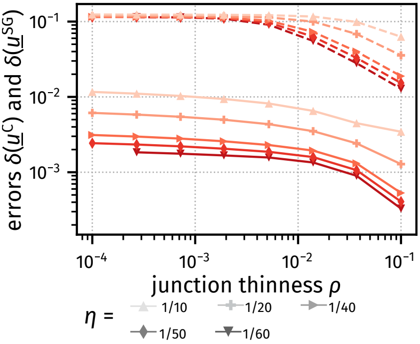
Comparison with full scale simulations

Influence of 3 parameters

- Scale ratio η
- Junction thinness ρ
- Microstructure relative position s







Comparison with full scale simulations

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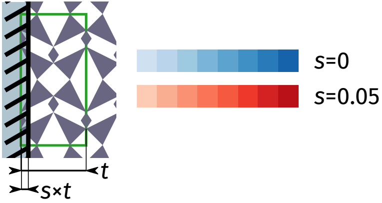

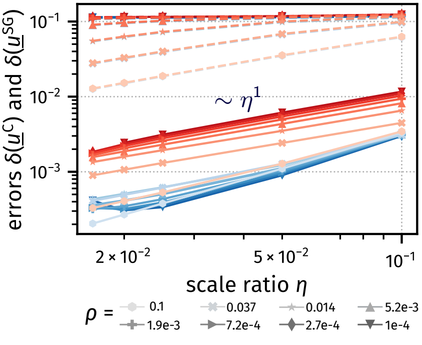

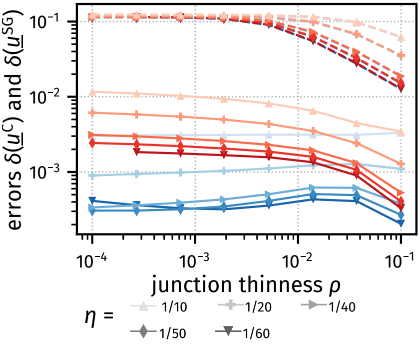

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