Continuum elasticity of Miura Tessellations

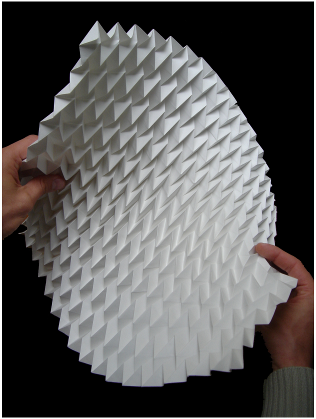
Hussein Nassar[1], Arthur Lebée[2], Laurent Monasse[3]

1: Dept. of Mechanical and Aerospace Engineering
University of Missouri, Columbia, USA

2: Laboratoire Navier (UMR CNRS 8205)
École des Ponts ParisTech; Université Gustave Eiffel; CNRS, France

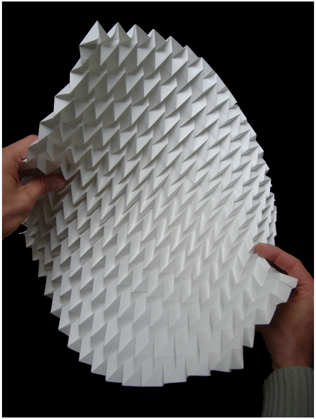
3: INRIA, Team Coffee and Laboratoire J.A. Dieudonné
Université Côte d'Azur, Nice, France

(Meta)-Surfaces from folded tessellations



Resch and Christiansen (1970)

(Meta)-Surfaces from folded tessellations



Resch and Christiansen (1970)

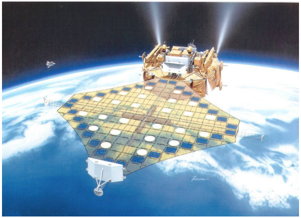
$\rightarrow$ what are the accessible shapes?

$\rightarrow$ what are the internal forces?

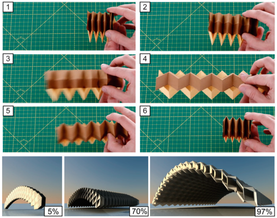
Motivations

Large deformations of a (micro)-structured surface?

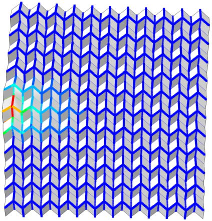
Deployable structures?/Morphing shells?/Meta-materials?



Miura (1993)



Filipov et al. (2015)



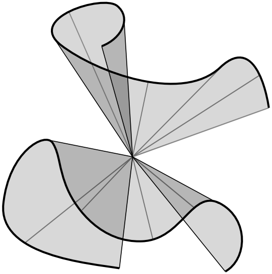
Grey et al. (2018)

$\rightarrow$ Equivalent elastic continuum?

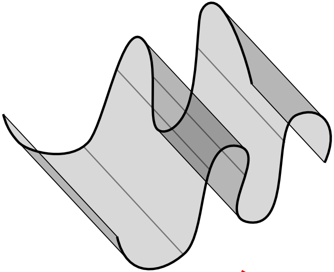
Contents

Smooth developable surfaces (Euler, 1772)

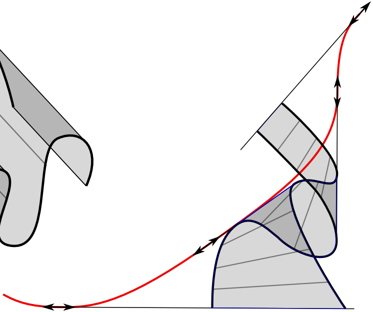
Unstretchable $\Leftrightarrow$ Gaussian curvature $K = 0$ $\Leftrightarrow$ Developable surface



Cone

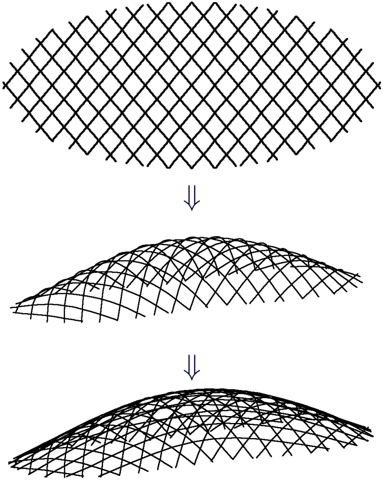


Cylinder



Tangent developable

Chebychev nets and Gridshells (Chebyshev, 1878)

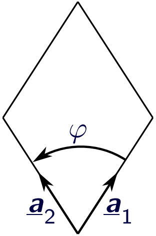


Douthe et al. (2007)

Metric:

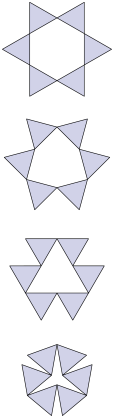
$$\begin{pmatrix} 1 & \cos\varphi \\ \cos\varphi & 1 \end{pmatrix}$$

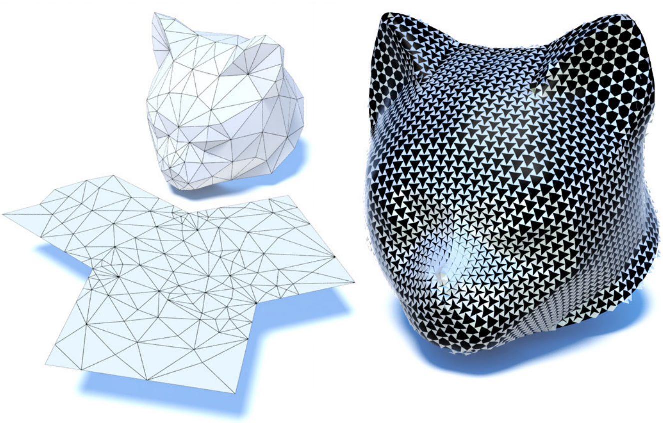
Free Curvature



Baverel et al. (2012)

Auxetic triangles

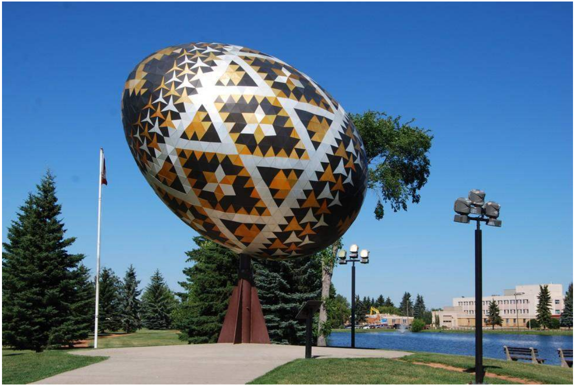




Conformal mappings! (Konaković-Luković et al., 2016)

Origami tessellations?

Resch and Christiansen (1970)

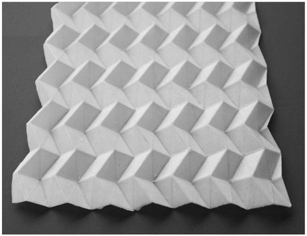


Vegreville (Canada)

"Resch and Christiansen expended six months of effort on this, but the sheet wouldn't conform to the egg shape." Blinn (1988)

Origami tessellations?

- Miura Ori:

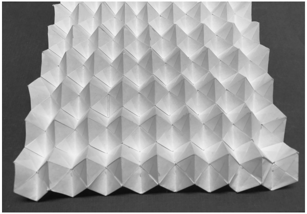


$$\nu_{\text{in-plane}} < 0$$

$$\nu_{\text{out-of-plane}} > 0$$

(Schenk and Guest, 2013; Wei et al., 2013)

- Eggbox pattern:



$$\nu_{\text{in-plane}} > 0$$

$$\nu_{\text{out-of-plane}} < 0$$

(Schenk, 2011; Nassar et al., 2017a,b)

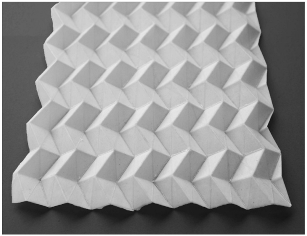
Origami tessellations?

- Miura Ori:

$$\nu_{\text{in-plane}} < 0$$

$$\nu_{\text{out-of-plane}} > 0$$

(Schenk and Guest, 2013; Wei et al., 2013)



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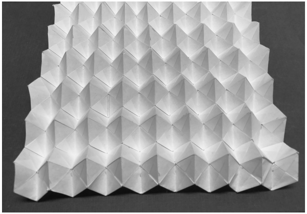
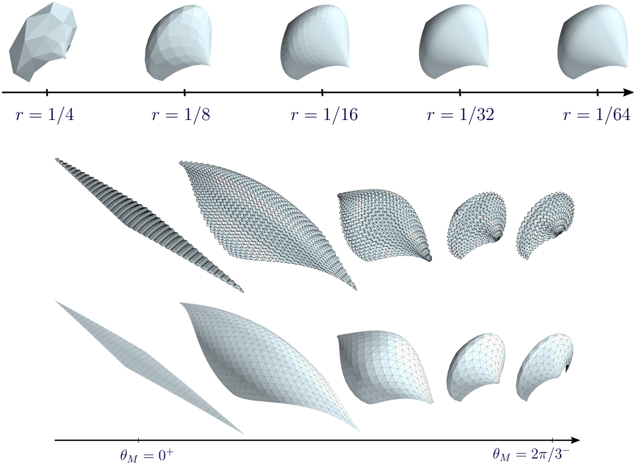
(Schenk, 2011; Nassar et al., 2017a,b)



Only small strains

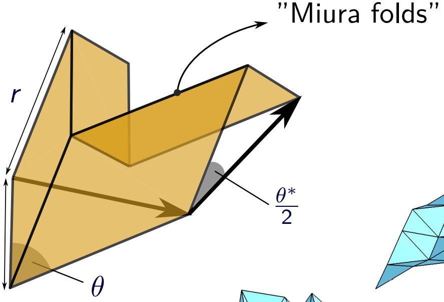
Some surfaces with the eggbox pattern (Nassar et al., 2017a)

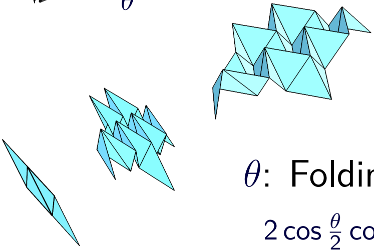


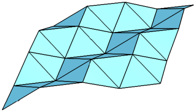
Nassar, Lebée, Monasse. *Curvature, metric and parametrization of origami tessellations: theory and application to the eggbox pattern.* Proceedings of the Royal Society A 2017

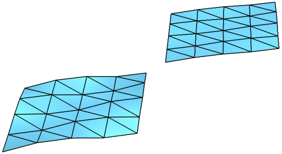
Contents

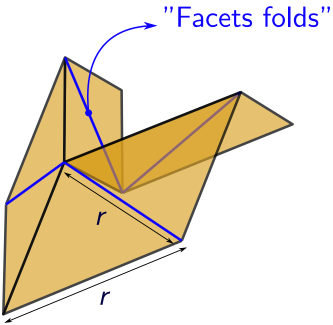
The flexible Miura ori







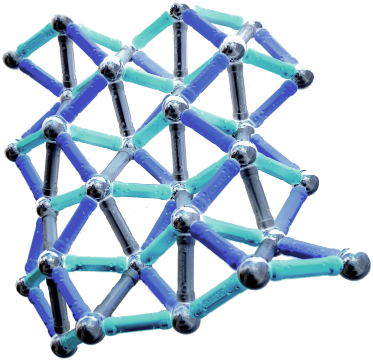




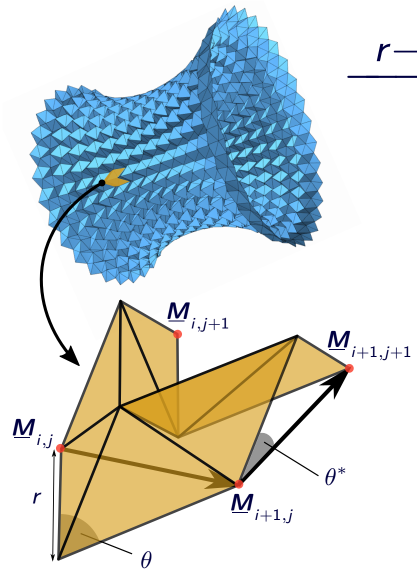
θ: Folding angle

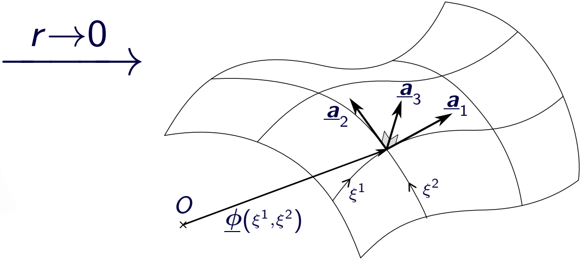
$$2\cos\frac{\theta}{2}\cos\frac{\theta^*}{2} = 1$$

Discrete modeling of the flexible Miura Ori



From discrete to continuous surfaces





Smooth $\underline{\phi}\,(\xi^1, \xi^2)$?

$\Rightarrow$ 1st & 2nd rank finite differences locally independent from $\underline{M}_{i,j}$

$\Rightarrow$ Constrains Metric and Curvature

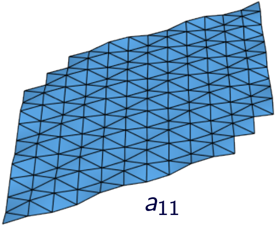
First rank (metric)

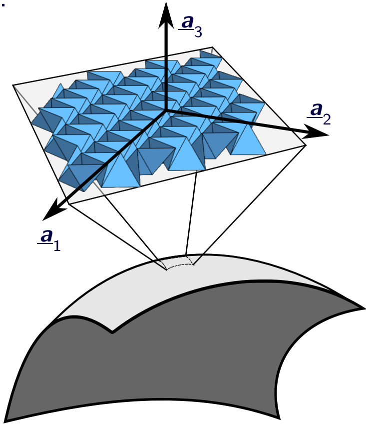
Planar periodic configurations at leading order:

$$\begin{cases} \underline{\boldsymbol{M}}_{i+1,j} - \underline{\boldsymbol{M}}_{i,j} \\ \underline{\boldsymbol{M}}_{i,j+1} - \underline{\boldsymbol{M}}_{i,j} \end{cases} \xrightarrow{r \to 0} \begin{cases} \partial_1 \underline{\phi} = \underline{\boldsymbol{a}}_1 \\ \partial_2 \underline{\phi} = \underline{\boldsymbol{a}}_2 \end{cases}$$

Metric:

$$a_{\alpha\beta}(\theta) = \underline{\boldsymbol{a}}_\alpha \cdot \underline{\boldsymbol{a}}_\beta = \begin{pmatrix} 4\sin^2\frac{\theta}{2} & 0 \\ 0 & 4\cos^2\frac{\theta^*_\theta}{2} \end{pmatrix}$$





Second rank (curvature and...)

Quadratic perturbations at second order:

$$\left\{\begin{array}{l} \underline{\boldsymbol{M}}_{i+1,j} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i-1,j} \\ \underline{\boldsymbol{M}}_{i,j+1} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i,j-1} \\ \underline{\boldsymbol{M}}_{i,j+1} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i+1,j} \end{array}\right. \xrightarrow{r\to 0} \left\{\begin{array}{l} \partial_{11}\underline{\phi} \\ \partial_{22}\underline{\phi} \\ \partial_{12}\underline{\phi} \end{array}\right., \text{ 9 components: } \Gamma^i_{\alpha\beta} = \underline{\boldsymbol{a}}^i \cdot \partial_{\alpha\beta}\underline{\phi}$$

Second rank (curvature and...)

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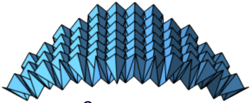
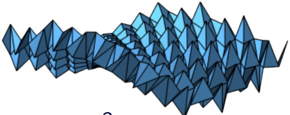
$$b_{\alpha\beta} = \underline{\boldsymbol{a}}^3 \cdot \partial_{\alpha\beta}\underline{\phi}$$



$\Gamma^3_{11} = b_{11}$



$\Gamma^3_{12} = b_{12}$

Curvature:

$$b_{\alpha\beta}(\theta) = \begin{pmatrix} N\cos^2\theta & M \\ M & -N\cos^2\theta^*_\theta \end{pmatrix}$$

Second rank (curvature and...)

Quadratic perturbations at second order:

$$\left\{\begin{array}{l} \underline{\boldsymbol{M}}_{i+1,j} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i-1,j} \\ \underline{\boldsymbol{M}}_{i,j+1} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i,j-1} \\ \underline{\boldsymbol{M}}_{i,j+1} - 2\underline{\boldsymbol{M}}_{i,j} + \underline{\boldsymbol{M}}_{i+1,j} \end{array}\right. \xrightarrow{r \to 0} \left\{\begin{array}{l} \partial_{11}\underline{\phi} \\ \partial_{22}\underline{\phi} \\ \partial_{12}\underline{\phi} \end{array}\right., \text{ 9 components: } \Gamma^i_{\alpha\beta} = \underline{\boldsymbol{a}}^i \cdot \partial_{\alpha\beta}\underline{\phi}$$

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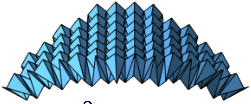
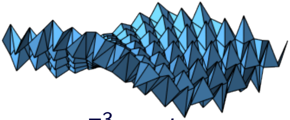
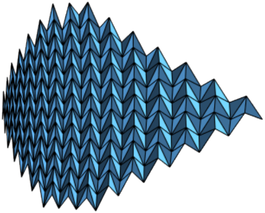


$\Gamma^3_{11} = b_{11}$

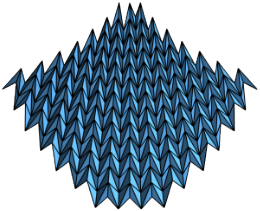


$\Gamma^3_{12} = b_{12}$

$$\Gamma^\gamma_{\alpha\beta} = \underline{\boldsymbol{a}}^\gamma \cdot \partial_{\alpha\beta}\underline{\phi}$$



Γ^1_{11}



Γ^1_{12}

The continuous fitting problem for the Miura ori

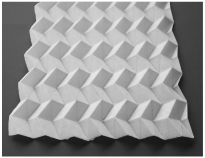
Find $\underline{\phi}$ and $\{\theta(\xi^\alpha), M(\xi^\alpha), N(\xi^\alpha)\}$ such that:

$$a_{\alpha\beta}(\theta) = \begin{pmatrix} 4\sin^2\theta & 0 \\ 0 & 4\cos^2\theta^*_\theta \end{pmatrix} \quad \text{and} \quad b_{\alpha\beta}(\theta) = \begin{pmatrix} N\cos^2\theta & M \\ M & -N\cos^2\theta^*_\theta \end{pmatrix}$$

Only if $\{\theta, M, N\}$ comply with Gauss-Codazzi-Mainardi equations!

Miura Ori:

$$\Rightarrow \quad \frac{\partial_{11}\underline{\phi}}{\cos^2\theta} + \frac{\partial_{22}\underline{\phi}}{\cos^2\theta^*_\theta} = \underline{\mathbf{0}}$$



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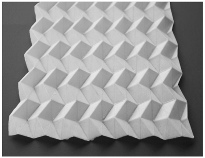
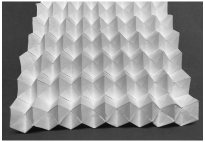


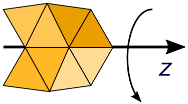
Eggbox Pattern:

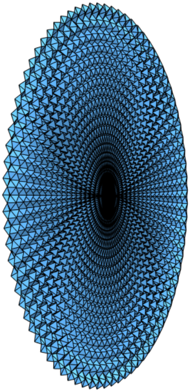
$$\Rightarrow \quad \frac{\partial_{11}\underline{\phi}}{\cos^2\theta} - \frac{\partial_{22}\underline{\phi}}{\cos^2\theta^*_\theta} = \underline{\mathbf{0}}$$



Some closed form illustrations of Miura surfaces

Axisymmetric I

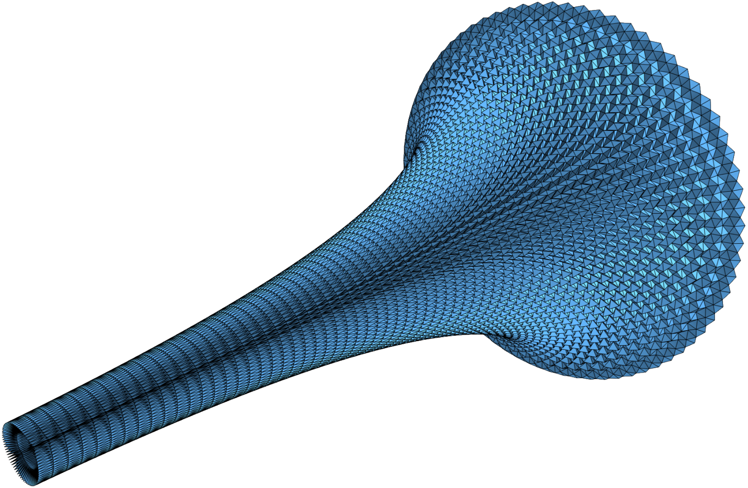




Const. neg. Gauss curvature

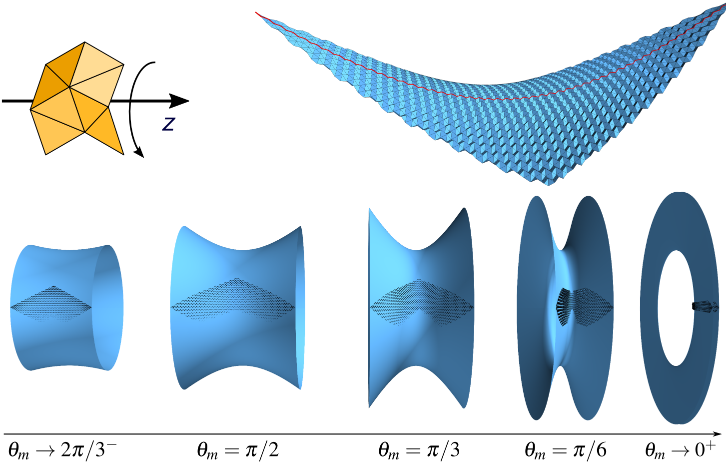
Some closed form illustrations of Miura surfaces

The pseudo sphere



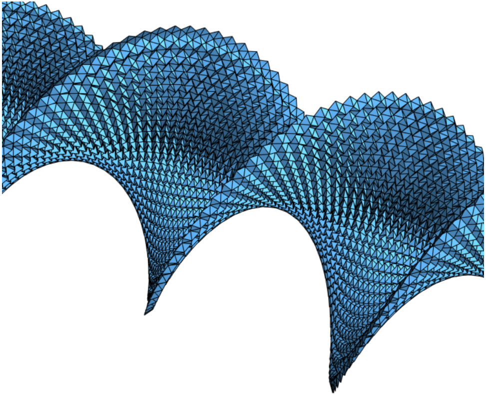
Some closed form illustrations of Miura surfaces

Miura Hyperboloid



Some closed form illustrations of Miura surfaces

Ruled surfaces



Helicoid

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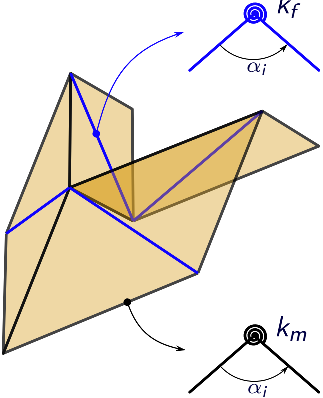
From discrete to continuous elasticity

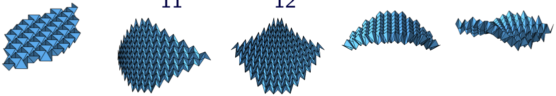


Strain energy density:

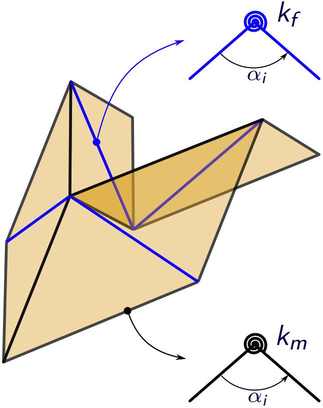
$$W^{\text{continuous}} = \frac{1}{r^2}\sum_i \frac{1}{2}k_i(\alpha_i - \alpha_i^0)^2$$

α_i are functions of:

a_{11} Γ^1_{11} Γ^1_{12} b_{11} b_{12}

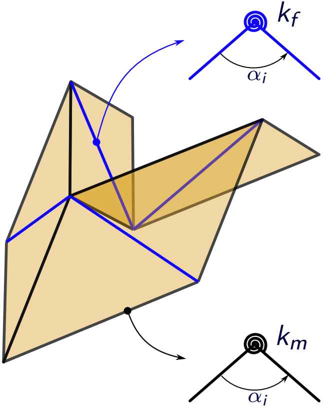


From discrete to continuous elasticity



At leading order when $r \to 0$:

From discrete to continuous elasticity



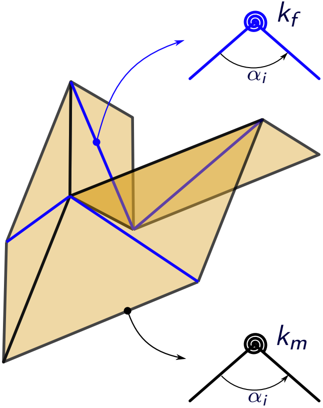
At leading order when $r \to 0$:

- if $k_m > 0$ and $k_f \sim k_m$:

$$W^{\text{continuous}}(a_{11})$$

$\rightarrow$ Membrane continuum

From discrete to continuous elasticity



At leading order when $r \to 0$:

- if $k_m > 0$ and $k_f \sim k_m$:

$$W^{\text{continuous}}(a_{11})$$

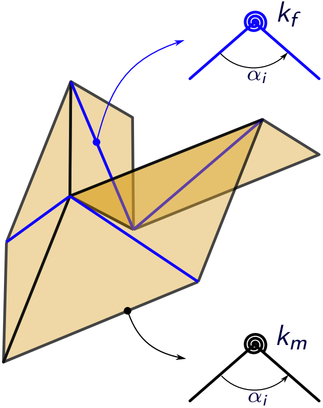
$\to$ Membrane continuum

- if $k_m = 0$ and $k_f > 0$:

$$W^{\text{continuous}}(\Gamma^1_{11}, \Gamma^1_{12}, b_{11}, b_{12}; a_{11})$$

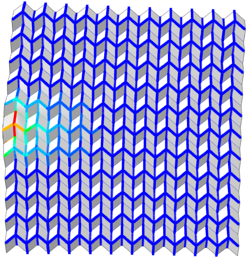
$\to$ Shell (out-of-plane) and Strain-gradient (in-plane) continuum

From discrete to continuous elasticity

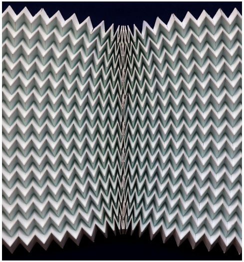


When $k_m > 0$ and $k_m \ll k_f$:

charac. length: $l \sim r\sqrt{\frac{k_f}{k_m}}$

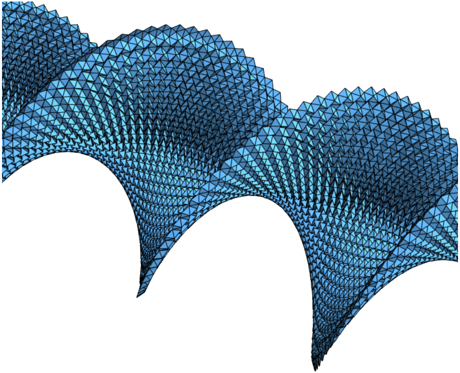


Grey et al. (2018)



Conclusions and outlooks

- Miura Ori surfaces
- 2 finite deformation continua:
  - Membrane
  - Shell + Strain-gradient



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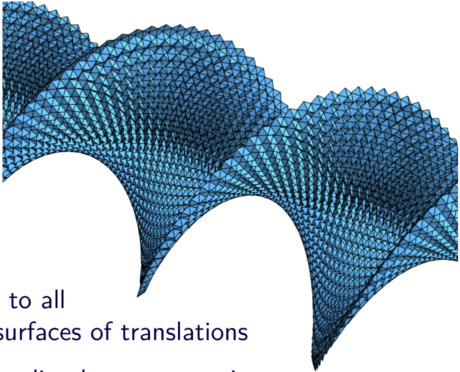

- Extended to all periodic surfaces of translations
- Strong coupling between metric and curvature?

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